A NOTE ON CLOSED MAPS AND COMPACT SETS

BY

E. MICHAEL

ABSTRACT

Some results relating closed maps to compact sets, which are already known for metrizable spaces, are here proved in a more general setting. Examples are given to indicate barriers to further improvements.

1. **Introduction.** The purpose of this note is to prove the following results.

THEOREM 1.1. *Let X be paracompact, Y locally compact or first-countable, and $f: X \to Y$ continuous and closed. Then $\partial f^{-1}(y)$ (the boundary of $f^{-1}(y)$) is compact for every $y \in Y$.* (1)

COROLLARY 1.2. *Let X be paracompact and $f: X \to Y$ continuous, closed and onto. Then every compact subset of Y is the image of some compact subset of X.*

For metrizable X, these results are known. In fact, in that case Theorem 1.1 was proved by Vainstein [9] for first countable Y, and hence follows for locally compact Y from a result of Arhangel'skii [1, Theorem 21] which implies that Y must then be metrizable. Corollary 1.2, for metrizable X, was announced by Arhangel'skii [2, Theorem 15].

For paracompact X, our results seem to be new, and Corollary 1.2 will be applied in [6].

Theorem 1.1 (in somewhat sharpened form) and Corollary 1.2 are proved in section 2, while section 3 contains some counterexamples to various plausible conjectures.

2. **Proofs.** We will obtain Theorem 1.1 as a consequence of a somewhat stronger result (Theorem 2.1). Let us call a point $y \in Y$ a *q-point* if it has a sequence of neighborhoods N_i such that, if $y_i \in N_i$ and the y_i are all distinct, then $y_1, y_2, \cdots$ has an accumulation point in Y. Call Y a *q-space* if every $y \in Y$ is a q-point. Clearly first-countable spaces and locally countably compact spaces are q-spaces; more generally, all p-spaces in the sense of A. Arhangel'skii [1] and all M-spaces in the sense of K. Morita [7] are q-spaces.

THEOREM 2.1. *Let $f: X \to Y$ be continuous, closed, and onto, where X is T_1. If $y \in Y$ is a q-point, then every continuous, real-valued function on X is bounded on $\partial f^{-}(y)$.*

Received October 30, 1964.

(1) That some restriction must be made on Y is seen by taking X to be the reals, Y the quotient space obtained from X by identifying all integers, and f the quotient map.

Proof. Suppose $h: X \to R$ is continuous, and unbounded on $\partial f^{-1}(y)$. Pick $x_1, x_2, \cdots$ in $\partial f^{-1}(y)$ such that

$$|h(x_{n+1})| > |h(x_n)| + 1.$$

Let

$$V_i = \{x \in X : |h(x) - h(x_i)| < 1/2\}.$$

Then V_i is open, $x_i \in V_i$, and $\{V_i\}_i$ is discrete (i.e. every $x \in X$ has a neighborhood intersectiong at most one V_i).

Let us now pick a sequence $z_i \in V_i \cap f^{-1}(N_i)$ (where N_i is as in the definition of q-point) such that all the $f(z_i)$ are distinct. This is easily done by induction: Let $z_1 = x_1$. Suppose suitable $z_1, \cdots, z_{i-1}$ have been found. Let

$$W_i = [V_i \cap f^{-1}(N_i^0)] - f^{-1}(\{f(z_2), \cdots, f(z_{i-1})\}),$$

and pick $z_i \in W_i - f^{-1}(y)$; this is possible, since W_i is open and $x_i \in W_i \cap \partial f^{-1}(y)$. This z_i clearly satisfies all requirements.

Let $Z = \{z_1, z_2, \cdots\}$. Since $z_i \in V_i$, every subset of Z is closed, and hence so is every subset of $f(Z)$. But $f(z_i) \in N_i$ and the $f(z_i)$ are all distinct, so that $f(Z)$ must have an accumulation point. This contradiction completes the proof.

COROLLARY 2.2. *If X is normal in Theorem 2.1, then $\partial f^{-1}(y)$ is countably compact.*

Proof. Suppose not. Then $\partial f^{-1}(y)$ has a subset $S = \{x_1, x_2, \cdots\}$ (all distinct) with no accumulation points in $\partial f^{-1}(y)$, and hence none in X because $f^{-1}(y)$ is closed (since f is closed). Define $g: S \to R$ by $f(x_n) = n$. Then g is continuous on the closed set S, and hence has a continuous extension $h: X \to R$. Since h is unbounded on $\partial f^{-1}(y)$, this contradicts Theorem 2.1.

Proof of Theorem 1.1. As observed above, first-countable spaces and locally compact spaces are q-spaces, so that every $\partial f^{-1}(y)$ is countably compact by Corollary 2.2. But $\partial f^{-1}(y)$ is closed in X, and hence paracompact. Since paracompact, countably compact spaces are compact [3], it follows that every $\partial f^{-1}(y)$ is compact.

Proof of Corollary 1.2. Without loss of generality, we may assume that Y is itself compact, and we must find a compact $C \subset X$ such that $f(C) = Y$. For all $y \in Y$, pick $p_y \in f^{-1}(y)$, and let

$$C_y = \partial f^{-1}(y) \qquad \text{if } \partial f^{-1}(y) \neq \phi$$

$$C_y = \{p_y\} \qquad \text{if } \partial f^{-1}(y) = \phi.$$

Let

$$C = \bigcup_{y \in Y} C_y,$$

and let $g = f | C$. Then $f(C) = Y$, and g is closed since C is closed in X. Also, for each $y \in Y$ we have $g^{-1}(y) = C_y$, so each $g^{-1}(y)$ is compact. Since Y is compact, this implies that $C = g^{-1}(Y)$ is compact (see, for instance, [5; Theorem 1]), and that completes the proof.

There is an analogue of Corollary 1.1 for *normal* X, obtained by everywhere changing "compact" to "countably compact"; the proof is based on Corollary 2.2. For *arbitrary* T_1-spaces X, the best I can do is to show (using Theorem 2.1) that every countably compact subset of Y is the image of a closed subset of X on which every continuous $f: X \to R$ is bounded. There ought to be something better.

3. Examples. The following examples indicate some limits to improving our results.

Example 3.1. *X is normal, Y is compact metric, and $f: X \to Y$ is continuous and closed, but some $\partial f^{-1}(y_0)$ is not compact.*

Let Y be the set of ordinals $\leqq \omega$, and let Z be the set of all countable ordinals, both with the order topology. Let $X = Y \times Z$, and let $f: X \to Y$ be the projection. Then X is normal; also X is countably compact, so f is closed. Finally, $\partial f^{-1}(\omega) = f^{-1}(\omega)$ is homeomorphic to Z, and is therefore not compact.

Examples 3.2. *X is completely regular, Y is compact metric and $f: X \to Y$ is continuous, closed and onto, but some $\partial f^{-1}(y_0)$ is not even pseudocompact.*(2) *Moreover, there is no countably compact $C \subset X$ such that $f(C) = Y$.*

Let R be the reals, N the integers, and $\bar{N}$ the closure of N in βR. Let

$$X = \beta R - (\bar{N} - N)$$

$$A = X - (R - N).$$

(This X seems to go back to Katětov, and is the space described in [4; p.97, 6P]). Let Y be the quotient space X/A, and $f: X \to Y$ the quotient map. Let $y_0 = f(A)$; then $\partial f^{-1}(y_0) = f^{-1}(y_0) = A$, and A is not pseudocompact since it contains N as an open-closed subset. It remains to check that Y is compact metric, which we do by showing that it is the one-point compactification of $R - N$ ($= X - A$). To do that, we must verify that every closed (in X) subset E of $R - N$ is compact. Since N and E are disjoint closed subsets of R, their closures $\bar{N}$ and $\bar{E}$ in βR are also disjoint. This implies that $\bar{E} \subset X$; since E is closed in X, we have $E = \bar{E}$, and hence E is compact.

Suppose, finally that $f(C) = Y$, and let us check that C is not countably compact. If $N \subset C$, then C is not countably compact since N has no limit point in C. If $n \notin C$ for some $n \in N$, then C is not countably compact either, because a sequence

(2) A space X is *pseudocompact* if every real-valued continuous function on it is bounded. Countably compact spaces are pseudocompact, and the converse is true in normal spaces.

in $R - N$ converging to A will then have no limit point in C. (Note that $C \supset R - N$). That completes the proof.

EXAMPLE 3.3. *X is normal, Y is compact Hausdorff, $f: X \to Y$ is continuous, closed and onto, but Y is not the image of any compact subset of X.*

Let X be the set of countable ordinals, with the order topology, and let A be the set of limit-ordinals in X. Let Y be the quotient space X/A, and $f: X \to Y$ the quotient map. Then X is normal and f is closed; if $f(C) = Y$ for some $C \subset X$, then $C \subset (X - A)$, so C is dense in X, and hence C cannot be compact (since X is not). To show that Y is compact metric, we need only check that it is the one-point compactification of $X - A$, and for that it suffices to prove that any closed (in X) subset S of $X - A$ is compact. But any such S must, in fact, be *finite* (since X is countably compact), and that completes the proof.

If the continuum hypothesis is assumed, then a recent example of M.E. Rudin [8] implies that "Hausdorff" can be strengthened to "metric" in Example 3.3:

EXAMPLE 3.4. (Assume continuum hypothesis). *X is normal, Y is compact metric, $f: X \to Y$ is continuous, closed, and onto, but Y is not the image of any compact subset of X.*

In [8; Example 2], M.E. Rudin constructs a normal, countably compact, non-compact space X, containing a dense open subset M which is separable metric and locally compact. Let $A = X - M$, let Y be the quotient space X/A, and let $f: X \to Y$ be the quotient map. Since every open $U \supset A$ has a compact complement in X, it follows that Y is essentially the one-point compactification of M, so that Y is compact metric. Since X is countably compact, f must be closed. If $f(C) = Y$ for some $C \subset X$, then $C \supset M$, hence C is dense in X, and therefore C cannot be compact (since X is not compact). That completes the proof.

REFERENCES

1. A. Arhangel'skii, *On a class of spaces containing all metric and all locally bicompact spaces*, Dokl. Akad. Nauk SSSR **151** (1963), 751–754 (= Soviet Math. Dokl. **4** (1963), 1051–1055).

2. A. Arhangel'skii, *Factor mappings of metric spaces*, Dokl. Akad. Nauk SSSR **155** (1964). (247–250). (=Soviet Math. Dokl. **5** (1964), 368–371.

3. R. Arens and J. Dugundji, *Remark on the concept of compactness*, Portugaliae Math. **9** (1950), 141–143.

4. L. Gillman and M. Jerison, *Rings of continuous functions*, van Nostrand (1960).

5. E. Halfar, *Compact mappings*, Proc. Amer. Math. Soc. **8** (1957), 828–830.

6. E. Michael, *$\aleph_0$-spaces*, to appear in J. Math. Mech.

7. K. Morita, *Products of normal spaces with metric spaces*, Math. Ann. **154** (1964), 365–382.

8. M. E. Rudin, *A technique for constructing examples*, to appear in Proc. Amer. Math. Soc.

9. I. A. Vainstein, *On closed mappings of metric spaces*, Dokl. Akad. Nauk SSSR **57** (1947), 319–321.

UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON